# Matrix averages relating to Ginibre ensembles

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**Abstract**
The theory of zonal polynomials is used to compute the average of a Schur
polynomial of argument $AX$, where $A$ is a fixed matrix and $X$ is from the
real Ginibre ensemble. This generalizes a recent result of Sommers and
Khoruzhenko (2009 *J. Phys. A: Math. Theor.* **42** 222002), and furthermore
allows analogous results to be obtained for the complex and real quaternion
Ginibre ensembles. As applications, the positive integer moments of the
general variance Ginibre ensembles are computed in terms of generalized
hypergeometric functions; these are written in terms of averages over matrices
of the same size as the moment to give duality formulas, and the averages
of the power sums of the eigenvalues are expressed as finite sums of zonal
polynomials.

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## 1. Introduction

The statistical properties of the eigenvalues of random $N \times N$ matrices with independent,
identically distributed (i.i.d.) real entries is a prominent topic in both physics and mathematics.
Physical applications began with the paper of May [26] on the stability of an ecological network
consisting of many components $\{y_i(t)\}_{i=1,\ldots,N}$, coupled in some unknown way. Suppose the
evolution of the components is governed by a first-order differential system, and write the
linearization of the latter about a fixed point in the form

$$\left[ \frac{\mathrm{d}\tilde{y}_i(t)}{\mathrm{d}t} \right]_{i=1,\ldots,N} = (-\mathbb{I}_N + B)[\tilde{y}_i(t)]_{i=1,\ldots,N}. \tag{1.1}$$

With the $N \times N$ matrix $B$ having all entries zero this differential equation exhibits
exponential relaxation to the fixed point. For general $B$, the system (1.1) is stable if and only
if all the eigenvalues of $B$ have a real part less than or equal to 1.

Since the coupling between components is unknown, it is reasonable to take the
components of $B$ to be random. May argued that for $B$ a dilute matrix (fraction $1 - c$

of its elements zero) with non-zero elements i.i.d. random variables having mean zero and variance $\sigma^2$, the spectral radius will be less than 1 provided $\sigma\sqrt{Nc} < 1$. This is an asymptotic result, requiring that $N$ be large. It is consistent with a rigorous result proved subsequently for the special case $c = 1$ [17].

In the mathematics literature, attention has been focussed not only on the spectral radius, but also on the eigenvalue density. With the elements standard Gaussians, by explicit calculation the eigenvalue density was proved to be asymptotically uniform in the disk of radius $\sqrt{N}$, centred about the origin in the complex plane [8]. This is the so called circle law [5, 19], which has recently been proved to remain true for general i.i.d. distributions [34]. Another mathematical property proved true in the Gaussian case by explicit calculation [9] is that for large $N$ the expected number of real eigenvalues is asymptotically $\sqrt{2N/\pi}$. Numerical evidence presented in [9] suggests that this result persists for general i.i.d. distributions of mean zero and unit variance although proof is yet to be found.

Of interest in both the physics and mathematics literature has been the integrability properties of eigenvalue distribution in the Gaussian case. Ginibre [18] was the first to seek an analytic formula for the eigenvalue probability density function (p.d.f.), giving rise to the name real Ginibre ensemble for real Gaussian matrices. In [18] an analytic expression was found for the eigenvalue p.d.f. conditioned so that all eigenvalues are real. For the conditioning specifying a general number of real eigenvalues, the analytic form of the eigenvalue p.d.f. was not obtained until the passing of a further 25 years [8, 24]. And it has not been until the past few years that analytic computations based on the eigenvalue p.d.f. have been mastered to the extent that the probability of a prescribed number of real eigenvalues can be calculated [23, 2, 29], and closed form expressions for the correlations obtained [7, 12, 13, 30, 32]. Furthermore, these analytic studies have been extended [3, 14] to the case of partially symmetric real Gaussian matrices [24].

A very recent result [31] relates to the average of the Schur polynomials $s_\kappa(\lambda_1, \ldots, \lambda_N)$ with respect to the eigenvalues $\{\lambda_j\}$ of matrices from the real Ginibre ensemble. We recall that the Schur polynomials are the basis for symmetric functions of $\{\lambda_j\}$, labelled by a partition $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_N \geqslant 0$ of non-negative integers, given by the ratio of determinants:

$$s_\kappa(\lambda_1, \ldots, \lambda_N) = \frac{\det\left[\lambda_j^{\kappa_k+N-k}\right]_{j,k=1,\ldots,N}}{\det\left[\lambda_j^{N-k}\right]_{j,k=1,\ldots,N}}. \tag{1.2}$$

With $\{\lambda_j\}$ being the eigenvalues of the matrix $X$, the Schur polynomials are often written as $s_\kappa(X)$. Making use of the knowledge of the explicit form of the eigenvalue p.d.f., it is proved in [31] that for $X$ a member of the real Ginibre ensemble

$$\langle s_\kappa(X)\rangle_X = \begin{cases} 2^{|\kappa|/2}\prod_{n=1}^{N}\dfrac{\Gamma((N-n+\kappa_n+1)/2)}{\Gamma((N-n+1)/2)}, & \text{all } \kappa_n \text{ even} \\ 0, & \text{otherwise,} \end{cases} \tag{1.3}$$

where $|\kappa| := \sum_{j=1}^{n}\kappa_j$.

It is the purpose of this paper to give a different viewpoint on this result. Explicitly, (1.3) will be deduced as a consequence of the theory of zonal polynomials. This viewpoint will allow for analogues of (1.3) to be given for the complex and real quaternion Ginibre ensemble. It further leads us to the evaluation of the moments of the characteristic polynomial in terms of generalized hypergeometric functions, in the case that the Gaussian matrices have a (matrix) distribution with a general variance. This in turn allows us to express the moments as different matrix integrals, in which the size of the matrix is equal to that of the moments, giving duality formulas. In the final section, again for the three Ginibre ensembles with a

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general variance, the averages of the power sums of the eigenvalues are expressed as finite sums of zonal polynomials.

## 2. Zonal polynomials

An alternative characterization of the Schur polynomials (1.3) is as the eigenfunctions of the differential operator:

$$\sum_{j=1}^{N}\left(\lambda_j\frac{\partial}{\partial\lambda_j}\right)^2 + \frac{N-1}{\alpha}\sum_{j=1}^{N}\lambda_j\frac{\partial}{\partial\lambda_j} + \frac{2}{\alpha}\sum_{1\leqslant j<k\leqslant N}\frac{\lambda_j\lambda_k}{\lambda_j-\lambda_k}\left(\frac{\partial}{\partial\lambda_j}-\frac{\partial}{\partial\lambda_k}\right) \tag{2.1}$$

in the case $\alpha=1$, with the structure

$$s_\kappa(\lambda_1,\ldots,\lambda_N) = m_\kappa + \sum_{\mu<\kappa}a_{\kappa\mu}m_\mu. \tag{2.2}$$

In (2.2), $m_\kappa$ denotes the monomial symmetric function indexed by the partition $\kappa$ (e.g. with $N=3$, $m_{1^2}=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3$), $\mu<\kappa$ refers to the dominance ordering on partitions specified by the requirement that $\sum_{j=1}^{l}\mu_j\leqslant\sum_{j=1}^{l}\kappa_j$ ($l=1,\ldots,N$) and the $a_{\kappa\mu}$ are scalars. For general $\alpha$ the eigenfunctions of (2.1) with the structure (2.2) are the symmetric Jack polynomials $P_\kappa^{(2/\alpha)}(\lambda_1,\ldots,\lambda_N)$.

Associated with a partition $\kappa$ is the generalized Pochhammer symbol

$$[u]_\kappa^{(\alpha)} = \prod_{j=1}^{N}\frac{\Gamma(u-(j-1)/\alpha+\kappa_j)}{\Gamma(u-(j-1)/\alpha)}, \tag{2.3}$$

and in terms of this and the classical Pochhammer symbol $(u)_n := u(u+1)\cdots(u+n-1)$ one defines

$$\begin{aligned} d_\kappa' &= \frac{\alpha^{|\kappa|}[(N-1)/\alpha+1]_\kappa^{(\alpha)}}{\bar{f}^{1/\alpha}(\kappa)}, \\ \bar{f}^{1/\alpha}(\kappa) &:= \prod_{1\leqslant i<j\leqslant N}\frac{(1+(j-i-1)/\alpha+\kappa_i-\kappa_j)_{1/\alpha}}{(1+(j-i-1)/\alpha)_{1/\alpha}}. \end{aligned} \tag{2.4}$$

The quantity $d_\kappa'$ in turn is used to define the renormalized Jack polynomials

$$C_\kappa^{(\alpha)}(\lambda_1,\ldots,\lambda_N) = \frac{\alpha^{|\kappa|}|\kappa|!}{d_\kappa'}P_\kappa^{(\alpha)}(\lambda_1,\ldots,\lambda_N). \tag{2.5}$$

In the cases $\alpha=2$, 1 and 1/2, the renormalized Jack polynomials are the so called zonal polynomials associated with the symmetric spaces $gl(N,\mathbb{R})/O(N)$, $gl(N,\mathbb{C})/U(N)$ and $u^*(2N)/Sp(2N)$ [25].

For present purposes, a key property of the zonal polynomials is their appearance on the right-hand sides of the matrix integrals [21, 25, 28]:

$$\langle s_\lambda(AO)\rangle_{O\in O(N)} = \begin{cases} \dfrac{C_\kappa^{(2)}(AA^T)}{C_\kappa^{(2)}((1)^N)}, & \lambda=2\kappa \\ 0, & \text{otherwise} \end{cases} \tag{2.6}$$

$$\langle s_\lambda(AU)s_\kappa(U^\dagger A^\dagger)\rangle_{U\in U(N)} = \delta_{\lambda,\kappa}\frac{C_\kappa^{(1)}(AA^\dagger)}{C_\kappa^{(1)}((1)^N)} \tag{2.7}$$

$$\langle s_\lambda(AS)\rangle_{S\in Sp(2N)} = \begin{cases} \dfrac{C_\kappa^{(1/2)}(AA^\dagger)}{C_\kappa^{(1/2)}((1)^N)}, & \lambda=\kappa^2, \\ 0, & \text{otherwise,} \end{cases} \tag{2.8}$$

where in (2.6) the partition $2\kappa$ is the partition obtained by doubling each part of $\kappa$, while in (2.8), $\kappa^2$ is the partition obtained by repeating each part of $\kappa$ twice. On the left-hand sides the averages are over the classical groups $O(N)$, $U(N)$, $Sp(2N)$ of unitary matrices with real, complex and real quaternion elements respectively, endowered with the corresponding Haar measure.

We immediately observe a structual similarity between (1.3) and (2.6). In fact, as will be shown in the next section, (2.6) implies and furthermore generalizes (1.3).

## 3. Averages over the Ginibre ensembles

That zonal polynomials are intimately related to averages over matrices with Gaussian entries is the theme of the monograph by Takemura [33]. This theme has further been developed in the works [20, 28]. More generally, zonal polynomials can be related to any measure $d\mu(X)$ on the space of random matrices with real, complex or real quaternion entries possessing the property of being invariant under the mappings $X \mapsto UX$, $X \mapsto XU$ for $U \in O(N)$, $U(N)$, $Sp(2N)$, respectively. With $\langle \cdot \rangle_X$ denoting an average in such a setting, and $\langle \cdot \rangle_U$ denoting the average over the corresponding classical group, to apply this theory to the Ginibre ensembles we first make note of a fundamental factorization property of the zonal polynomials with respect to the former, namely (see e.g. [25])

$$\left\langle C_\kappa^{(\alpha)}(AU^\dagger BU) \right\rangle_U = \frac{C_\kappa^{(\alpha)}(a_1, \ldots, a_N)C_\kappa^{(\alpha)}(b_1, \ldots, b_N)}{C_\kappa^{(\alpha)}((1)^N)}. \tag{3.1}$$

Here $C_\kappa^{(\alpha)}((1)^N) := C_\kappa^{(\alpha)}(x_1, \ldots, x_N)|_{x_1 = \cdots = x_N = 1}$ and $\{a_i\}$, $\{b_i\}$ are the eigenvalues of $A$, $B$, respectively.

**Proposition 1** [33]. *One has*

$$\left\langle C_\kappa^{(\alpha)}(AXBX^\dagger) \right\rangle_X = \frac{C_\kappa^{(\alpha)}(A)C_\kappa^{(\alpha)}(B)}{\left(C_\kappa^{(\alpha)}((1)^N)\right)^2} \left\langle C_\kappa^{(\alpha)}(XX^\dagger) \right\rangle_X. \tag{3.2}$$

**Proof.** For any $f(X)$ integrable with respect to $d\mu(X)$, the invariance of $d\mu(X)$ under $X \mapsto UX$ tells us that

$$\langle f(AXBX^\dagger) \rangle_X = \langle \langle f(AUXBX^\dagger U^\dagger) \rangle_U \rangle_X, \tag{3.3}$$

while the invariance under $X \mapsto XU$ gives

$$\langle f(AXBX^\dagger) \rangle_X = \langle \langle f(AXUBU^\dagger X^\dagger) \rangle_U \rangle_X. \tag{3.4}$$

Choosing $f = C_\kappa^{(\alpha)}$ in (3.3) and using (3.1) shows

$$\left\langle C_\kappa^{(\alpha)}(AXBX^\dagger) \right\rangle_X = \frac{C_\kappa^{(\alpha)}(A)}{C_\kappa^{(\alpha)}((1)^N)} \left\langle C_\kappa^{(\alpha)}(XBX^\dagger) \right\rangle_X. \tag{3.5}$$

Choosing $f = C_\kappa^{(\alpha)}$ in (3.4) and again using (3.1) allows the right-hand side of (3.5) to be evaluated, and (3.2) results.                                                         □

We can use (3.2) combined with (2.6)–(2.8) to obtain a generalization of (1.3) for each of the real, complex and real Ginibre ensembles. First we consider the case of a general measure $d\mu(X)$ invariant under multiplication by unitary matrices.

**Proposition 2.** *Let $d\mu(X)$ be as required for (3.2). For real matrices*

$$\langle s_\mu(AX)\rangle_X = \begin{cases} \dfrac{C_\kappa^{(2)}(AA^T)}{\left(C_\kappa^{(2)}((1)^N)\right)^2}\langle C_\kappa^{(2)}(XX^T)\rangle_X, & \mu = 2\kappa \\ 0, & \text{otherwise;} \end{cases} \tag{3.6}$$

*for complex matrices*

$$\langle s_\mu(AX)s_\kappa(X^\dagger A^\dagger)\rangle_X = \delta_{\lambda,\kappa}\frac{C_\kappa^{(1)}(AA^\dagger)}{\left(C_\kappa^{(1)}((1)^N)\right)^2}\langle C_\kappa^{(1)}(XX^\dagger)\rangle_X; \tag{3.7}$$

*for real quaternion matrices*

$$\langle s_\mu(AX)\rangle_X = \begin{cases} \dfrac{C_\kappa^{(1/2)}(AA^T)}{\left(C_\kappa^{(1/2)}((1)^N)\right)^2}\langle C_\kappa^{(1/2)}(XX^\dagger)\rangle_X, & \mu = \kappa^2 \\ 0, & \text{otherwise.} \end{cases} \tag{3.8}$$

**Proof.** Consider first (3.6). It follows from (2.6) that

$$\langle s_\mu(AX)\rangle_X = \langle\langle s_\mu(AXO)\rangle_O\rangle_X = \begin{cases} \dfrac{\langle C_\kappa^{(2)}(A^T AXX^T)\rangle_X}{C_\kappa^{(2)}((1)^N)}, & \mu = \kappa^2 \\ 0, & \text{otherwise.} \end{cases}$$

Making use of (3.2) in the case $\alpha = 2$ gives (3.6). The derivation of (3.7) and (3.8) begins with (2.7) and (2.8), then makes use of (3.2) in an analogous manner. $\square$

We see from (3.6) and (2.3) that (1.3) is reclaimed if we can show that for $d\mu(X) \propto e^{-(1/2)\text{Tr } XX^T}(dX)$,

$$\frac{1}{C_\kappa^{(2)}((1)^N)}\langle C_\kappa^{(2)}(XX^T)\rangle_X = 2^{|\kappa|/2}[N/2]_\kappa^{(2)}. \tag{3.9}$$

For this we change variables $XX^T = A$, using the result $(dX) \propto (\det A)^{-1/2}(dA)$ (see e.g. [11, equation (3.30)]), to obtain

$$\langle C_\kappa^{(2)}(XX^T)\rangle_X = \frac{1}{C}\int_{A>0} e^{-(1/2)\text{Tr } A}(\det A)^{-1/2}C_\kappa^{(2)}(A)(dA), \tag{3.10}$$

where $C$ is such that the rhs equals unity when $\kappa = 0^N$, and $A > 0$ denotes that the integral is over the space of positive definite matrices. Changing variables now to the eigenvalues and eigenvectors of $A$ shows that (3.10) is proportional to

$$\int_0^\infty d\lambda_1 \cdots \int_0^\infty d\lambda_N \prod_{l=1}^N \lambda_l^{-1/2} e^{-\lambda_l/2} C_\kappa^{(2)}(\lambda_1, \ldots, \lambda_N) \prod_{j<k}^N |\lambda_k - \lambda_j|. \tag{3.11}$$

We recognize the integral in (3.11) as appearing in the integration formula (see [11, equation (12.152)]):

$$\frac{1}{C}\int_0^\infty d\lambda_1 \cdots \int_0^\infty d\lambda_N \prod_{l=1}^N \lambda_l^a e^{-\lambda_l} C_\kappa^{(\alpha)}(\lambda_1, \ldots, \lambda_N) \prod_{j<k}^N |\lambda_k - \lambda_j|^{2/\alpha}$$
$$= C_\kappa^{(\alpha)}((1)^N)[a + (N-1)/\alpha + 1]_\kappa^{(\alpha)} \tag{3.12}$$

(take $\alpha = 2$, $a = -1/2$ and change variables $t_l \mapsto t_l/2$), which itself is a limiting case of a generalization of the Selberg integral conjectured by Macdonald and proved by Kadell and Kaneko (see [15] and references therein), and (3.9) follows.

Proceeding similarly, making use of (3.12) for $\alpha = 1$, we can show that for $X$ complex with $d\mu(X) \propto e^{-\text{Tr}(XX^\dagger)}(dX)$,

$$\frac{1}{C_\kappa^{(1)}((1)^N)}\langle C_\kappa^{(1)}(XX^\dagger)\rangle_X = [N]_\kappa^{(1)}. \tag{3.13}$$

And for $X$ real quaternion with $d\mu(X) \propto e^{-\text{Tr}(XX^\dagger)}(dX)$—the trace now being with respect to the quaternion structure and so selecting only one of the diagonal elements from each $2 \times 2$ block—we have

$$\frac{1}{C_\kappa^{(1/2)}((1)^N)}\langle C_\kappa^{(1/2)}(XX^\dagger)\rangle_X = 2^{-|\kappa|}[2N]_\kappa^{(2)}. \tag{3.14}$$

Substituting (3.10), (3.13) and (3.14) in proposition 2 gives the sought generalization of (1.3) for the Ginibre ensembles.

**Corollary 1.** *With the distribution of the real Ginibre ensemble proportional to $e^{-(1/2)\text{Tr }XX^T}$, and the distribution of the complex and real quaternion ensembles proportional to $e^{-\text{Tr}(XX^\dagger)}$, one has for the real Ginibre ensemble*

$$\langle s_\mu(AX)\rangle_X = \begin{cases} \dfrac{2^{|\kappa|}[N/2]_\kappa^{(2)}}{C_\kappa^{(2)}((1)^N)}C_\kappa^{(2)}(AA^T), & \mu = 2\kappa \\ 0, & \text{otherwise}; \end{cases} \tag{3.15}$$

*for the complex Ginibre ensemble*

$$\langle s_\mu(AX)s_\kappa(X^\dagger A^\dagger)\rangle_X = \delta_{\mu,\kappa}\frac{[N]_\kappa^{(1)}}{C_\kappa^{(1)}((1)^N)}C_\kappa^{(1)}(AA^\dagger); \tag{3.16}$$

*and for the real quaternion Ginibre ensemble*

$$\langle s_\mu(AX)\rangle_X = \begin{cases} \dfrac{2^{-|\kappa|}[2N]_\kappa^{(1/2)}}{C_\kappa^{(1/2)}((1)^N)}C_\kappa^{(1/2)}(AA^\dagger), & \mu = \kappa^2 \\ 0, & \text{otherwise}. \end{cases} \tag{3.17}$$

The fact that [11, proposition 12.23]

$$P_\kappa^{(\alpha)}((1)^N) = \frac{\alpha^{|\kappa|}[N/\alpha]_\kappa^{(\alpha)}}{h_\kappa} \tag{3.18}$$

where [11, proposition 12.28] $1/h_\kappa$ is the coefficient of $(x_1 + \cdots + x_N)^{|\kappa|}$ in $P_\kappa^{(\alpha)}(x)$ allows the results of corollary 1 to be written as

$$\langle s_\mu(AX)\rangle_X = \begin{cases} h_\kappa P_\kappa^{(2)}(AA^T), & \mu = 2\kappa \\ 0, & \text{otherwise}; \end{cases}$$

$$\langle s_\mu(AX)s_\kappa(X^\dagger A^\dagger)\rangle_X = \delta_{\mu,\kappa}h_\kappa P_\kappa^{(1/2)}(AA^\dagger)$$

$$\langle s_\mu(AX)\rangle_X = \begin{cases} h_\kappa P_\kappa^{(1/2)}(AA^\dagger), & \mu = \kappa^2 \\ 0, & \text{otherwise}, \end{cases}$$

respectively. In this form corollary 1 appears in the unpublished manuscript by Rains [28].

## 4. Hypergeometric functions

We know from workings in [6] that

$$\frac{2^{|\kappa|}[N/2]_{\kappa}^{(2)}}{C_{\kappa}^{(2)}((1)^N)} = \frac{1}{|\kappa|!2^{|\kappa|}}d'_{2\kappa}|_{\alpha=1}$$

$$\frac{[N]_{\kappa}^{(1)}}{C_{\kappa}^{(1)}((1)^N)} = \frac{1}{|\kappa|!}(d'_{\kappa}|_{\alpha=1})^2 \qquad (4.1)$$

$$\frac{2^{-|\kappa|}[2N]_{\kappa}^{(1/2)}}{C_{\kappa}^{(1/2)}((1)^N)} = \frac{1}{2^{|\kappa|}|\kappa|!}d'_{\kappa^2}|_{\alpha=1},$$

which we substitute into the results of corollary 1 as appropriate. To see the consequence of this, we recall the definition of the generalized hypergeometric functions

$$_pF_q^{(\alpha)}(a_1, \ldots, a_p, b_1, \ldots, b_q; x_1, \ldots, x_N) := \sum_{\kappa} \frac{1}{|\kappa|!} \frac{[a_1]_{\kappa}^{(\alpha)} \cdots [a_p]_{\kappa}^{(\alpha)}}{[b_1]_{\kappa}^{(\alpha)} \cdots [b_q]_{\kappa}^{(\alpha)}} C_{\kappa}^{(\alpha)}(x_1, \ldots, x_N).$$

$$a \qquad (4.2)$$

For $p = 0, q = 1$, this is a generalization of the binomial expansion and we have [11, equation (13.4)]

$$_1F_0^{(\alpha)}(a; x_1, \ldots, x_N) = \prod_{j=1}^{N}(1 - x_j)^{-a}. \qquad (4.3)$$

It follows from this that upon multiplying both sides of (3.15) and (3.17) by $[-r]_{\mu}^{(1)}/d'_{\mu}|_{\alpha=1}$ and both sides of (3.16) by $[-r]_{\mu}^{(1)}[-r]_{\kappa}^{(1)}/d'_{\mu}|_{\alpha=1}d'_{\kappa}|_{\alpha=1}$, and summing over $\mu$ (or $\mu$ and $\kappa$ in the case of (3.16)), the left-hand sides of each of the identities can be summed according to (4.3). For the resulting matrix averages to be well defined, we require $r \in \mathbb{Z}_{\geqslant 0}$. To simplify the right-hand sides, we note from (2.3) that

$$[u]_{2\kappa}^{(1)} = 2^{2|\kappa|}[u/2]_{\kappa}^{(2)}[(u+1)/2]_{\kappa}^{(2)}, \qquad [u]_{\kappa^2}^{(1)} = [u]_{\kappa}^{(1/2)}[u-1]_{\kappa}^{(1/2)},$$

and then make use of (4.2). Consequently we obtain the following set of matrix integral evaluations.

**Corollary 2.** Let $r \in \mathbb{Z}_{\geqslant 0}$. One has, for the real, complex and real quaternion Ginibre ensembles, respectively,

$$\langle \det(\mathbb{I}_N - AX)^r \rangle_X = {}_2F_0^{(2)}(-r/2, (-r+1)/2; 2AA^T)$$

$$\langle \det(\mathbb{I}_N - AX)^r \det(\mathbb{I}_N - X^{\dagger}A^{\dagger})^r \rangle_X = {}_2F_0^{(1)}(-r, -r; AA^{\dagger}) \qquad (4.4)$$

$$\langle \det(\mathbb{I}_{2N} - AX)^r \rangle_X = {}_2F_0^{(1/2)}(-r, -r-1; AA^{\dagger}/2).$$

Suppose $A$ in (4.4) is invertible, and write $\Sigma = (A^{\dagger}A)^{-1}$. We can then rewrite (4.4) to read

$$\langle \det(\mathbb{I}_N - xX)^r \rangle_X = {}_2F_0^{(2)}(-r/2, (-r+1)/2; 2x^2\Sigma)$$

$$\langle \det(\mathbb{I}_N - xX)^r \det(\mathbb{I}_N - \bar{x}X^{\dagger})^r \rangle_X = {}_2F_0^{(1)}(-r, -r; |x|^2\Sigma) \qquad (4.5)$$

$$\langle \det(\mathbb{I}_{2N} - xX)^r \rangle_X = {}_2F_0^{(1/2)}(-r, -r+1; |x|^2\Sigma/2).$$

Here the averages are over $N \times N$ matrices $X$ with real, complex and real quaternion elements, respectively, having a distribution proportional to $e^{-\text{Tr}(XX^T\Sigma^{-1})/2}$ in the real case and $e^{-\text{Tr}(XX^{\dagger}\Sigma^{-1})}$ in the complex and quaternion real cases. We remark that in the case where

$\Sigma$ equals the identity, results from [1, 4, 22] allow the averages in the complex and real quaternion cases to be expressed as a determinant and Pfaffian, respectively. One would expect the average in the real case, with $\Sigma = \mathbb{I}_N$, to similarly be expressible as a Pfaffian, although the details remain to be worked out.

Introducing $Y = X - (1/x)\mathbb{I}_N$ and setting $W = YY^\dagger$, we then have that $W$ is distributed as a non-central Wishart distribution (see e.g. [27]) in the first case, and its complex and real quaternion generalization in the other two cases. Furthermore, for $r = 2s$ even in the real case, and for general non-negative integer $r$ in the complex and real quaternion cases, the determinants in (4.5) can be written entirely in terms of $W$, and we obtain

$$|x|^{2Ns}\langle(\det W)^s\rangle_W = {}_2F_0^{(2)}(-s, -s + 1/2; 2x^2\Sigma)$$

$$|x|^{2Nr}\langle(\det W)^r\rangle_W = {}_2F_0^{(1)}(-r, -r; |x|^2\Sigma) \tag{4.6}$$

$$|x|^{Nr}\langle(\det W)^{r/2}\rangle_W = {}_2F_0^{(1/2)}(-r, -r - 1; |x|^2\Sigma/2).$$

The identities (4.6) are noteworthy for the fact that in the real case a different generalized hypergeometric function evaluation is known [27, theorem 10.3.7],

$$\langle(\det W)^s\rangle_W = (\det \Sigma)^s 2^{Ns}\frac{\Gamma_m((N/2 + s))}{\Gamma_m(N/2)}{}_1F_1^{(2)}(-s; N/2; -\Sigma^{-1}/2x^2), \tag{4.7}$$

where

$$\Gamma_m(u) := \pi^{m(m-1)/2}\prod_{j=1}^m \Gamma(u - (i - 1)/2).$$

This implies (after some minor simplification) the identity between generalized hypergeometric functions

$${}_2F_0^{(2)}(-s, -s + 1/2; Y) = [N/2]_{s^N}^{(2)}(\det Y)^s {}_1F_1^{(2)}(-s; N/2; -Y^{-1}). \tag{4.8}$$

Using the property of Jack polynomials [11, exercises 12.1 q.1 and 2]

$$(\det Y)^s P_\kappa^{(\alpha)}(Y^{-1}) = P_{\kappa^s}^{(\alpha)}(Y)$$

where $\kappa^s := (s - \kappa_N, s - \kappa_{N-1}, \dots, s - \kappa_1)$, noting from (2.4) that

$$\frac{2^{|\kappa|}}{2^{|\kappa^s|}}\frac{d'_{\kappa^s}}{d'_\kappa} = \frac{[(N + 1)/2]_{\kappa^s}^{(2)}}{[(N + 1)/2]_\kappa^{(2)}}$$

and using the property of the generalized Pochhammer symbol (2.3)

$$[u]_{\kappa^s}^{(\alpha)} = (-1)^{|\kappa^s|}\frac{[(N - 1)/\alpha - u + 1 - s]_{s^N}^{(\alpha)}}{[(N - 1)/\alpha - u + 1 - s]_\kappa^{(\alpha)}}$$

(a consequence of the functional equation for the gamma function), (4.8) can be verified directly by comparing coefficients of $P_\kappa^{(2)}(Y)$ on both sides.

## 5. Duality identities

There are many matrix ensembles of $N \times N$ matrices $\{X\}$ for which $\langle\det(\mathbb{I}_N - xX)^r\rangle_X$ can be expressed in terms of an average over dual matrix ensembles where the size of the matrices is $r \times r$ (see e.g. [11]). An example of relevance to the present study is an identity of Fyodorov and Khoruzhenko [16] (see also [10]), which reads

$$\langle|\det(z\mathbb{I}_N - AU)|^{2p}\rangle_{U \in U(N)} \propto \int_0^\infty dt_1 \cdots \int_0^\infty dt_p$$

$$\times \prod_{l=1}^p \frac{\det(|z|^2\mathbb{I}_p + t_l AA^\dagger)}{(1 + t_l)^{N+2p}} \prod_{1 \leqslant j < k \leqslant p} |t_k - t_j|^2. \tag{5.1}$$

To write the right-hand side as a matrix average, we require a result from random matrix theory [11, exercises 3.6 q.3] giving that the matrix $Y = X^\dagger (B^{-1/2})^\dagger B^{-1/2} X$, where $X$ is an $(N + p) \times p$ standard complex Gaussian matrix, and $A$ is a complex Wishart matrix $B = b^\dagger b$ with $b$, and $(N + p) \times (N + p)$ standard complex Gaussian matrix, has eigenvalue p.d.f. proportional to

$$\prod_{l=1}^{p} \frac{1}{(1 + t_l)^{N+2p}} \prod_{1 \leqslant j < k \leqslant p} |t_k - t_j|^2.$$

Noting too that

$$\prod_{l=1}^{p} \det(|z|^2 \mathbb{I}_p + t_l A A^\dagger) = \det(|z|^2 \mathbb{I}_{Np} + Y \otimes A A^\dagger)$$

we see that (5.1) can be rewritten as the matrix average duality:

$$\langle |\det(z\mathbb{I}_N - AU)|^{2p} \rangle_{U \in U(N)} = \langle \det(|z|^2 \mathbb{I}_{Np} + Y \otimes A A^\dagger) \rangle_Y. \tag{5.2}$$

The close relationship seen in section 3 between averages over the unitary group which contain an arbitrary matrix, and averages over the complex Ginibre ensemble, suggests that analogous formulas hold for the averages in (4.5). This is indeed the case. The sought identities follow from a representation for $_2F_0^{(\alpha)}(-r, -a/\alpha - (r-1); Y), r \in \mathbb{Z}^+$ as an $r$-dimensional integral, valid for all $a > 0$.

**Proposition 3.** *Let $r \in \mathbb{Z}^+, a > 0$ and $Y$ be an $N \times N$ matrix. We have*

$$_2F_0^{(\alpha)}(-r, -a/\alpha - (r-1); Y) = \frac{1}{\widetilde{W}_{a-1,2\alpha,r}} \int_0^\infty dt_1 \cdots \int_0^\infty dt_r$$

$$\times \prod_{l=1}^{r} e^{-t_l} t_l^{a-1} \det(\mathbb{I}_N + (t_l/\alpha)Y) \prod_{1 \leqslant j < k \leqslant r} |t_k - t_j|^{2\alpha}, \tag{5.3}$$

*where*

$$\widetilde{W}_{\lambda_1,\beta,n} := \int_0^\infty dt_1 \cdots \int_0^\infty dt_n \prod_{l=1}^{n} e^{-t_l} t_l^{\lambda_1} \prod_{1 \leqslant j < k \leqslant n} |t_k - t_j|^{\beta}$$

*(this normalization is a well-known limiting case of the Selberg integral, and as such can be evaluated as a product of gamma functions [11, proposition 4.7.3] although we do not need this fact).*

**Proof.** With $\kappa'$ denoting the conjugate partition, obtained by interchanging the rows and columns of the diagram of $\kappa$, we have [11, exercises 12.4 q.2]

$$[u]_{\kappa'}^{(\alpha)} = (-\alpha)^{-|\kappa|}[-\alpha u]_{\kappa}^{(1/\alpha)}.$$

Also, from the definition of $d'_\kappa$ in terms of arm and leg lengths [11, equation (12.37)], and the corresponding definition of $h_\kappa$ [11, equation (12.58)], one sees

$$d'_{\kappa'} = \alpha^{|\kappa|} h_\kappa |_{\alpha \mapsto 1/\alpha}.$$

Recalling (2.5) and (4.2) it follows that

$$_2F_0^{(\alpha)}(-r, -a/\alpha - (r-1); x_1, \ldots, x_N) = \sum_\kappa \frac{\alpha^{-2|\kappa|}[r\alpha]_\kappa^{(1/\alpha)}[a + (r-1)\alpha]_\kappa^{(1/\alpha)}}{h_\kappa |_{\alpha \mapsto 1/\alpha}} P_{\kappa'}^{(\alpha)}(X).$$

$$\tag{5.4}$$

The significance of this series form, as distinct from the form implied by (4.2), is that the coefficient of $P_{\kappa'}^{(\alpha)}(X)$ permits the integral representation

$$\frac{\alpha^{-|\kappa|}[r\alpha]_\kappa^{(1/\alpha)}[a+(r-1)\alpha]_\kappa^{(1/\alpha)}}{h_\kappa|_{\alpha \mapsto 1/\alpha}} = \frac{1}{\widetilde{W}_{a-1,2\alpha,r}} \int_0^\infty dt_1 \cdots \int_0^\infty dt_r$$

$$\times \prod_{l=1}^r t_l^{a-1}\, e^{-t_l} P_\kappa^{(1/\alpha)}(T) \prod_{1 \leqslant j < k \leqslant r} |t_k - t_j|^{2\alpha},$$

which is just a rewrite of (3.12), after making use of (3.18). Substituting in (5.4), and recalling the dual Cauchy identity for Jack polynomials [11, equation (12.186)]

$$\prod_{k,l=1}^N (1 + x_k y_l) = \sum_\kappa P_\kappa^{(\alpha)}(X) P_{\kappa'}^{(1/\alpha)}(Y),$$

(5.3) follows.                                                                                                      □

**Corollary 3.** *Let the averages over X be as in (4.5). We have*

$$\langle \det(x\mathbb{I}_N - X)^{2s}\rangle_X = \frac{1}{\widetilde{W}_{0,4,r}} \int_0^\infty dt_1 \cdots \int_0^\infty dt_s$$

$$\times \prod_{l=1}^s e^{-t_l} \det(x^2\mathbb{I}_N + t_l\Sigma) \prod_{1 \leqslant j < k \leqslant s} (t_k - t_j)^4 \tag{5.5}$$

$$\langle |\det(x\mathbb{I}_N - X)|^{2r}\rangle_X = \frac{1}{\widetilde{W}_{0,2,r}} \int_0^\infty dt_1 \cdots \int_0^\infty dt_r$$

$$\times \prod_{l=1}^r e^{-t_l} \det(|x|^2\mathbb{I}_N + t_l\Sigma) \prod_{1 \leqslant j < k \leqslant r} (t_k - t_j)^2 \tag{5.6}$$

$$\langle \det(x\mathbb{I}_{2N} - X)^r\rangle_X = \frac{1}{\widetilde{W}_{0,1,r}} \int_0^\infty dt_1 \cdots \int_0^\infty dt_r$$

$$\times \prod_{l=1}^r e^{-t_l} \det(|x|^2\mathbb{I}_{2N} + t_l\Sigma) \prod_{1 \leqslant j < k \leqslant r} |t_k - t_j|. \tag{5.7}$$

Each of the right-hand sides in the above identities can be written as matrix averages involving Wishart matrices. For example, with $Y = a^\dagger a$, where $a$ is an $r \times r$ matrix of standard complex Gaussian entries, (5.6) can be written as

$$\langle |\det(x\mathbb{I}_N - X)|^{2r}\rangle_X = \langle \det(|x|^2\mathbb{I}_{2Nr} + Y \otimes \Sigma)\rangle_Y. \tag{5.8}$$

In the theory of the complex Ginibre ensemble with $\Sigma = \mathbb{I}_N$, it is well known that the eigenvalue density $\rho_{(1)}((x, y))$ (with $N \mapsto N + 1$ for convenience) satisfies

$$\rho_{(1)}((x, y)) = \frac{e^{-|z|^2}}{\pi N!} \langle |\det(z\mathbb{I}_N - X)|^2\rangle_X \tag{5.9}$$

(see e.g. [16, equation (1.3)]). Here $z = x + \mathrm{i}y$ and $\{X\}$ is the complex Ginibre ensemble with $\sigma = \mathbb{I}_N$. Even though the equality is restricted to $\Sigma = \mathbb{I}_N$, this motivates us to use the duality identity (5.6) to investigate the right-hand side of (5.9) when $\{X\}$ is the complex Ginibre ensemble for a more general variance matrix. In particular, suppose $\Sigma = \mathrm{diag}(\sigma, (1)^{N-1})$. Then (5.6) gives that the right-hand side of (5.9) equals

$$\frac{\sigma}{\pi N!} \int_{|z|^2}^{\infty} e^{-t} t^N \, dt + \frac{(1-\sigma)|z|^2}{\pi N!} \int_{|z|^2}^{\infty} e^{-t} t^{N-1} \, dt. \tag{5.10}$$

For $0 < |z|^2 < N$ and $N$ large the leading behaviour is

$$\frac{\sigma}{\pi} + \frac{(1-\sigma)|z|^2}{\pi N},$$

while for $|z| = \sqrt{N} - r$ and $\sigma$ fixed, for large $N$ we obtain

$$\frac{1}{2\pi}(1 + \mathrm{erf}(\sqrt{2}r))$$

independent of $\sigma$. Since the boundary of the eigenvalue support is $|z| = \sqrt{N}$, this suggests that our choice of $\Sigma$ did not affect the largest eigenvalues. On the other hand, if we were to set $\sigma \mapsto \sigma\sqrt{N}$, as well as $|z| = \sqrt{N} - r$, the asymptotic form of (5.9) is dependent on $\sigma$, indicating that then the largest eigenvalues have been altered.

## 6. Power sum averages

In the previous section we computed, in (4.5), the integer moments of the characteristic polynomials for real, complex and real quaternion Gaussian matrices with a general variance matrix $\Sigma$. Here we average the power sums $p_k(X) := \mathrm{Tr}(X^k)$ over the same class of matrices. For real Gaussian matrices with $\Sigma = \mathbb{I}_N$, this average was computed using (3.15) in the case $A = \mathbb{I}_N$ [31]. If we make use of each of the identities of corollary 1 for general $A$, together with the expansion [25, Ex. 1.4.10]

$$p_k(X) = \sum_{l=0}^{k-1} (-1)^l s_{(k-l,1^l)}(X),$$

we can generalize the result of [31].

**Corollary 4.** *Let the averages over $X$ be as in (4.5). We have*

$$\langle p_k(X) \rangle_X = \begin{cases} 0, & k \text{ odd} \\ 2^{k/2} \displaystyle\sum_{l=0}^{k/2-1} \frac{[N/2]^{(2)}_{(k/2-l,1^l)}}{C^{(2)}_{(k/2-l,1^l)}((1)^N)} C^{(2)}_{(k/2-l,1^l)}(\Sigma), & k \text{ even} \end{cases}$$

$$\langle p_k(X) p_k(X^\dagger) \rangle_X = \sum_{l=0}^{k-1} \frac{[N]^{(1)}_{(k-l,1^l)}}{C^{(1)}_{(k-l,1^l)}((1)^N)} C^{(1)}_{(k-l,1^l)}(\Sigma)$$

$$\langle p_k(X) \rangle_X = \begin{cases} 0, & k \text{ odd or } k/2 > N \\ 2^{-k/2} \dfrac{[2N]^{(1/2)}_{1^{k/2}}}{C^{(1/2)}_{1^{k/2}}((1)^N)} C^{(1/2)}_{1^{k/2}}(\Sigma), & k \leqslant 2N \text{ even.} \end{cases}$$

Let $\rho^{\mathrm{r}}_{(1)}$ and $\rho^{\mathrm{c}}_{(1)}((x,y))$ denote the density of the real and (upper half plane) complex eigenvalues for real Gaussian matrices with variance matrix $\Sigma$. Then we have

$$\langle p_k(X) \rangle_X = \int_{-\infty}^{\infty} x^k \rho^{\mathrm{r}}_{(1)}(x) \, dx + \int_{\mathbb{R}^2_+} ((x+\mathrm{i}y)^k + (x-\mathrm{i}y)^k) \rho^{\mathrm{c}}_{(1)}((x,y)) \, dx \, dy,$$

where $\mathbb{R}^2_+ := \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}^+\}$. The average in corollary 4 in the complex case can be rewritten as an average involving both the one and two point correlations, which we refrain from writing down. In the real quaternion case, the average again can be written in terms of

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just the one point density. Here there are no real eigenvalues, and we have

$$\langle p_k(X) \rangle_X = \int_{\mathbb{R}^2_+} ((x + \mathrm{i}y)^k + (x - \mathrm{i}y)^k) \rho^{\mathrm{c}}_{(1)}((x, y)) \, \mathrm{d}x \, \mathrm{d}y.$$

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